Improved Stability Criterion for Lubrication Flow between Counterrotating Rollers

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Free surface flows due to counterrotating rollers or a roller near a plate lead to the formation of ribbing lines on the surface under certain conditions. Following the pioneering work of Pitts and Greiller (1961), Savage (1977) carried out a stability analysis of such flows. He obtained a necessary algebraic condition for the formation of ribbing lines. He also carried out a numerical study which provides sufficient conditions for onset of these instabilities. However, this method is cumbersome and sometimes subject to numerical errors, since it involves trial and error and integration of a differential equation to infinity. The procedure involves assuming a value for N (number of lines per unit width), solving the resulting initial value problem to infinity, and comparing the value with the boundary value available at infinity. The process is repeated until a satisfactory value of N is obtained. However, though simple to use, the algebraic condition is too conservative. This paper presents a simple necessary condition that gives results that are tighter than the old condition of Savage (1977). These conditions also give bounds on the number of ribbing lines.

DERIVATION OF NECESSARY CONDITIONS

For the geometry and coordinate system shown in Figure 1, the stability analysis leads to the following problem for the dimensionless perturbation pressure G (Savage, 1977). The analysis requires lubrication theory conditions to prevail.

$$G_{XX} + \frac{3H_X}{H} G_X - N^2 G = 0 \qquad (1)$$

$$G(-\infty) = 0$$

$$G(C) = -a + bN^2 < 0$$

$$G_X(C) = -d < 0$$

where

$$a = (1-\alpha) - \frac{\sqrt{2C}}{3\beta N_{Ca}}\sqrt{\frac{h_o}{R}}$$

$$b = \frac{1}{6N_{Ca}}\left(\frac{h_o}{R}\right)(1 + C^2)^2$$

$$d = \frac{2C\alpha}{1 + C^2}$$

$$H(X) = (1 + X^2)$$

$$g(x) = (3\mu U/h^2(C))G(X)$$

$$n(2Rh_o)^{1/2} = N, h(x) = h_o H(X)$$

and α, β are functions of N_{Ca}.

Now let

$$-\eta^2 \le \frac{3H_X}{H} \le \eta^2.$$

In the case of the two-roller geometry $\eta^2 = 3.0$. The constant η^2 is similar to a Lipschitz constant; depending on the form of $H(X)$, it will be different for other geometries. We now construct barrier functions bounding $G(X)$ from above and below. Note that $G(X)$ attains its maximum absolute value at $X = C$ and that $G, G_X \le 0$ on $-\infty < X \le C$ (Savage, 1977).

Consider the function F satisfying

$$F_{XX} + \eta^2 F_X - N^2 F = 0 \qquad (2)$$

$$F(-\infty) = 0, \quad F(C) = G(C)$$

Subtracting Eq. 2 from Eq. 1 and setting $u = G - F$

$$u_{XX} - N^2 u = \frac{-3H_X G_X}{H} + \eta^2 F_X$$

$$\le -\eta^2 G_X + \eta^2 F_X$$

$$\le -\eta^2 u_X$$

At an extremum

$$u_{XX} - N^2 u \le 0 \text{ or } u_{XX} \le N^2 u.$$

It can be shown easily that, since $u = 0$ on the boundary and because the above inequality holds, $u \ge 0$ on $-\infty < X \le C$, i.e., $G \ge F$ everywhere in $-\infty < X \le C$.

Hence,

$$|G_X(C)| \ge |F_X(C)|$$

since G and F are negative in $-\infty < X \le C$.

But since

$$F(X) = G(C)\exp\left[\frac{(-\eta^2 + \sqrt{\eta^4 + 4N^2})}{2}(X - C)\right]$$

$$|G_X(C)| \ge \left|\frac{-\eta^2 + \sqrt{\eta^4 + 4N^2}}{2} G(C)\right| \qquad (3)$$

By a similar argument it can be shown that

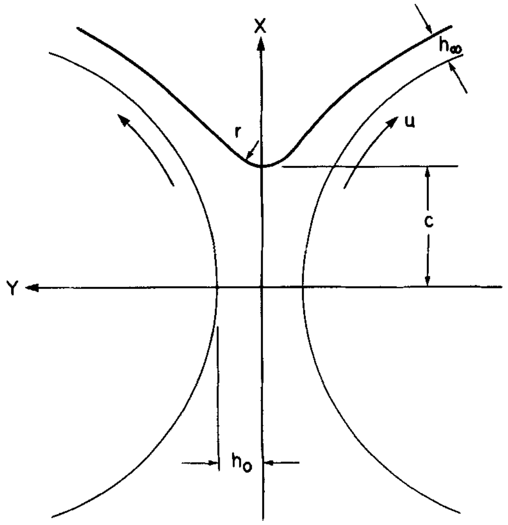

Figure 1. Geometry and coordinate system for roll coating device.

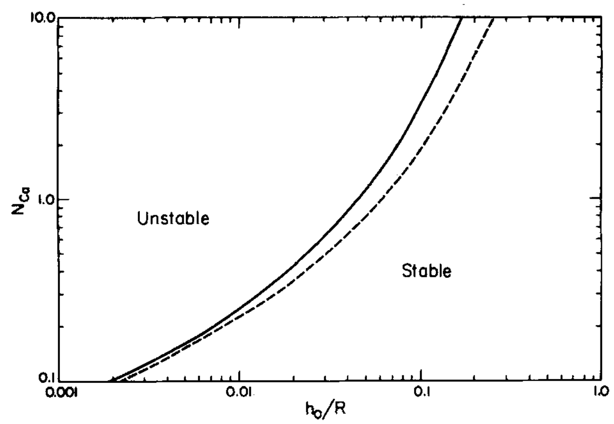

Figure 2. Stability conditions from Eqs. 8 and 9 (solid line) and the condition described by Eq. 10 (dotted line).

$$|G_X(C)| \le \left|\frac{\eta^2 + \sqrt{\eta^4 + 4N^2}}{2} G(C)\right| \quad (4)$$

Substituting for $G(C), G_X(C)$ we get from Eqs. 3 and 4

$$b^2N^6 - 2abN^4 + (a^2 + bd\eta^2)N^2 - (d^2 + d\eta^2 a) \le 0 \quad (5)$$

and

$$b^2N^6 - 2abN^4 + (a^2 - db\eta^2)N^2 + (d\eta^2 a - d^2) \ge 0 \quad (6)$$

Equations 5 and 6 are the required necessary conditions for the existence of a solution to Eq. 1. They provide bounds on the possible values that N can take and also necessary conditions for the onset of ribbing instabilities.

Using the fact that $g(c) = -a + bN^2 \le 0$, i.e., $N^2 \le a/b$, we get from Eq. 6

$$-2abN^4 + [a^2 - db\eta^2]N^2 + \left[da\eta^2 - d^2 + \frac{a^3}{b}\right] \ge 0 \quad (7)$$

Inequality (Eq. 7) will have no real roots if the coefficients

$$a^2 - db\eta^2 \le 0 \quad (8)$$

and

$$da\eta^2 - d^2 + \frac{a^3}{b} \le 0 \quad (9)$$

Equations 8 and 9 can be used to predict the possible onset of ribbing. The results are compared (Figure 2) with the criterion of Savage (1977) which amounts to the necessary condition

$$a \le 0 \quad (10)$$

The region to the left of the curves is possibly unstable. The solid line shows the results from calculations carried out by using Eqs. 8 and 9, which give much better results than the dotted line (criterion 10). Both conditions are necessary conditions; however, the new condition reduces the region of possible instability.

SUMMARY OF RESULTS

Barrier functions on the perturbation pressures in lubrication flows leading to ribbing instabilities have been obtained. These lead to an improved necessary condition (Eqs. 8 and 9) for onset of ribbing lines. The condition is algebraic and simple to use for other geometries.

NOTATION

C = dimensionless position of meniscus
$G(X)$ = dimensionless pressure perturbation
$H(X)$ = dimensionless form of half the distance between rollers
N = $(2Rh_o)^{1/2}n$
N_{Ca} = capillary number $\mu U/\sigma$
R = radius of roller
U = speed of roller surface
X = dimensionless x coordinate $x/(2Rh_o)^{1/2}$
c = position of meniscus
$g(x)$ = pressure perturbation
$h(x)$ = half the distance between rollers
x = distance along x axis
α = $h_\infty/h(c)$ parameter depending on N_{Ca}
β = $r/h(c)$ parameter depending on N_{Ca}
η^2 = max $|3H_X/H|$
μ = viscosity

LITERATURE CITED

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Manuscript received December 8, 1981; revision received April 12, and accepted May 28, 1982.